Dec 2019

Roll No. [ ] Total No. of Pages : 02

Total No. of Questions : 18

B.Tech. (CE/ME/ECE/EE) (2018 & Onward) (Sem.–1)
B.Tech. (Agriculture Engineering)/(Automation & Robotics)
/(Automobile Engineering)/(CSE)/(Electrical & Electronics Engineering)/(Electronics & Electrical Engineering)

MATHEMATICS-I

Subject Code : BTAM-101-18

M.Code : 75353

Time : 3 Hrs. Max. Marks : 60

INSTRUCTIONS TO CANDIDATES :

1. SECTION-A is COMPULSORY consisting of TEN questions carrying TWO marks each.
2. SECTION - B & C have FOUR questions each.
3. Attempt any FIVE questions from SECTION B & C carrying EIGHT marks each.
4. Select atleast TWO questions from SECTION - B & C.

SECTION-A

1. Test the convergence of the following series $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\dots$
2. State the Raabe's test.
3. State Rolle's theorem.
4. State Langrange's mean value theorem.
5. Prove that $\int_0^{\frac{\pi}{2}} \log \tan x\, dx = 0.$
6. Evaluate $\int_0^1 \int_0^x e^{\frac{x}{y}}\, dy dx.$
7. Change the order of integration of $\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x,y)\, dx dy.$
8. Find the first order derivative of $z = x^3 + y^3 - 3axy.$
9. Find the rank of the following matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$

Q7 For the series $\sum_{n=1}^{\infty} \frac{nx^n}{4^n(n^2+1)}$ Find:

a) For what values of x does the series converge absolutely?

b) For what values of x does the series converge conditionally?

c) Find the interval of convergence.

Q8. Determine whether the given system of equations is consistent or not? if consistent solve it

$$x+2y-z=3, 3x-y+2z=1, 2x-2y+3z=2, x-y+z=-1$$

Q9. Find the eigen values and eigen vectors of the following matrix :

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

1

Solved Question Paper

B.tech (CE/ME/ECE/EE)

Btam-101-18 (DEC 2019)

MCode-75353

Q1. Test the convergence of the following series

$$\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\cdots$$

Solⁿ $\frac{2!}{3}+\frac{3!}{3^2}+\frac{4!}{3^3}+\cdots$

$$\sum a_n=\sum\frac{(n+1)!}{3^n}$$

$$a_{n+1}=\frac{(n+2)!}{3^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+2)!}{3^{n+1}}}{\frac{(n+1)!}{3^n}}\right|$$

$$=\lim_{n\to\infty}\frac{\cancel{(n+2)!}\;\cancel{3^n}}{(n+1)\;\cancel{(n+2)!}\;\cancel{3^n}\,3}$$

$$=\lim_{n\to\infty}\frac{1}{3(n+1)}=\frac{1}{\infty}=0$$

By Ratio test,

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=l$ and $l<1$ then $\sum_{n=1}^{\infty}a_n$ converges

So, ~~as~~ Here $l=0<1$ so, It converges.

Q 2.) Statement of Raabe's test?

Solⁿ ⇒ The series $\sum U_n$ of Positive terms is

(i) Convergent If $\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) > 1$

(ii) divergent If $\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) < 1$

Q.3) State Rolle's theorem

Solⁿ If a function f is

(i) Continous in closed interval $[a,b]$

(ii) derivable in open interval (a,b)

(iii) $f(a) = f(b)$

then there exist atleast one real no. c

$c \in (a,b)$ and $f'(c) = 0$

Q4) State Langrange's mean value theorem

Solⁿ If a function f is

(i) Continous in $[a,b]$

(ii) derivable in (a,b)

$\exists . c \in (a,b)$ then there exist atleast a c

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

5.) Prove that $\int_0^{\pi/2} \log \tan x \, dx = 0$

Solⁿ $\int_0^{\pi/2} \log \tan x \, dx$ — (1)

Using Property $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_0^{\pi/2} \log\left(\tan\left(\frac{\pi}{2}-x\right)\right) dx$$

$$I = \int_0^{\pi/2} \log \cot x \, dx \quad \text{— (2)}$$

Add (1) & (2)

$$2I = \int_0^{\pi/2} (\log \tan x + \log \cot x)\, dx$$

$$2I = \int_0^{\pi/2} \log \tan x \cot x \, dx$$

$$2I = \int_0^{\pi/2} \log \tan x \frac{1}{\tan x} \, dx$$

$$2I = \int_0^{\pi/2} \log 1 \, dx \qquad [\because \log 1 = 0]$$

$$\boxed{I = 0}$$

(8). Evaluate $\int_0^1 \int_0^x e^{y/x} \, dy \, dx$

Sol^n

$$\int_0^1 \int_0^x \left[e^{y/x} \right] dy \, dx$$

$$\int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^x dx$$

$$\int_0^1 x \left[e^{x/x} - e^{0/x} \right] dx$$

$$\int_0^1 x \, (e-1) \, dx$$

$$(e-1) \left[\frac{x^2}{2} \right]_0^1$$

$$\frac{(e-1)}{2}$$

$\cancel{V = e^{x/y}}$

$\cancel{dV = e^{x/y} \cdot \frac{1}{y} \, dy}$

$\cancel{x = 0 \Rightarrow V = 1}$

(i) Change the order of integration of

$$\int_0^1 \int_{y^2}^{y^{1/3}} f(x,y)\, dx\, dy$$

Sol^n $y=0,\ y=1$

$y^2 = x,\ y^{1/3} = x \Rightarrow x^3 = y$

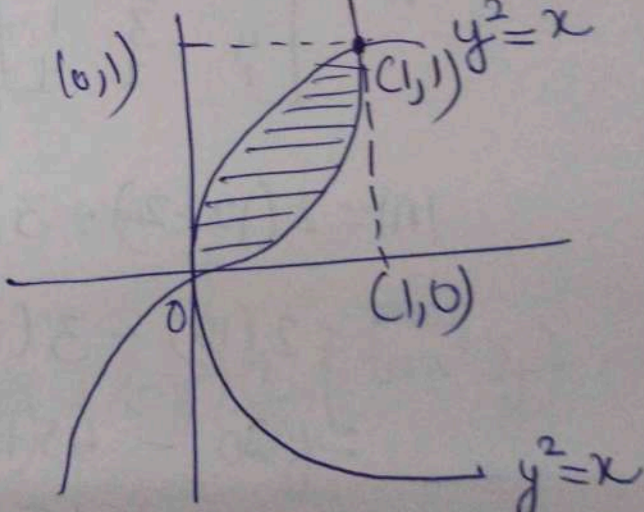
Points of intersection are

$(0,0)$ and $(1,1)$

After changing the order of

integration

$$\int_0^1 \int_{y^2}^{y^{1/3}} f(x,y)\, dx\, dy$$



$$= \int_0^1 \int_{x^3}^{\sqrt{x}} f(x,y)\, dx\, dy$$

(8) Find the first order derivative of

$$Z = x^3 + y^3 - 3axy.$$

Sol^n

$$z = x^3 + y^3 - 3axy$$

$$\frac{dz}{dx} = 3x^2 + 3y^2\frac{dy}{dx} - 3a\left[1.y + x\frac{dy}{dx}\right]$$

$$\frac{dz}{dx} = 3x^2 + 3y^2\frac{dy}{dx} - 3ay - 3ax\frac{dy}{dx}$$

$$\frac{dz}{dx} = 3x^2 - 3ay + \frac{dy}{dx}\left[3y^2 - 3ax\right]$$

(9) Find the rank of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A| = 2(12-2) - 3(16-1) + 4(8-3)$$

$$= 2(10) - 3(15) + 4(5)$$

$$= 20 - 45 + 20$$

$$= 40 - 45 = -5 \neq 0$$

$\rho(A)$ = order of matrix $= 3$

$$\rho(A) = 3$$

Find the determinant of following of Matrix

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 6 \\ 1 & 4 & 7 \end{bmatrix}$$

Sol^n Determinant $\Rightarrow$

$$|A| = 1(21-24) - 2(14-6) + 5(8-3)$$
$$= 1(-3) - 2(8) + 5(5)$$
$$= -3 - 16 + 25$$
$$= -19 + 25 = \underline{\underline{6}} \text{ Ans.}$$

(11) If $u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$. Show that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

Sol^n.

$$u = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}$$

$$\frac{\partial u}{\partial y} = x^2 \frac{\partial}{\partial y}\left(\tan^{-1}\frac{y}{x}\right) + (-y^2)\frac{\partial}{\partial y}\left(\tan^{-1}\frac{x}{y}\right)$$
$$+ \tan^{-1}\frac{x}{y} \cdot \frac{\partial}{\partial y}(-y^2)$$

$$= x^2 \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} - y^2 \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} + \tan^{-1}\frac{x}{y} - 2y$$

$$\frac{\partial u}{\partial y} = \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$\& \ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x^3+xy^2}{x^2+y^2} - 2y \cdot \tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x(x^2+y^2)}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y \tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y \cdot \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y}$$

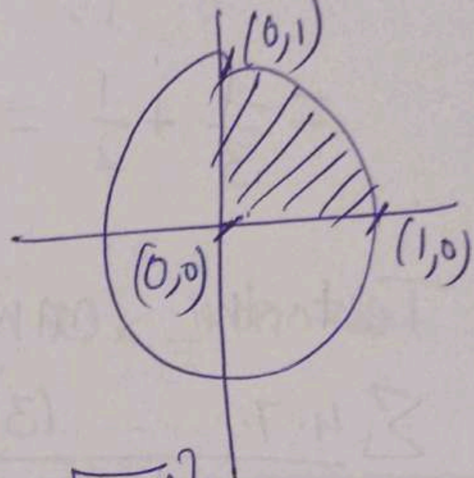
$$= 1 - \frac{2y^2}{x^2+y^2}$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

1) Evaluate $\iint \frac{xy}{(1-y^2)^{1/2}}\,dx\,dy$ over the first quadrant of the circle $x^2+y^2=1$

Sol^n equation of circle is

$x^2+y^2=1$.



The region of integration is given by.

$\{(x,y): (0\le x\le 1 \text{ and } 0\le y\le \sqrt{1-x^2}\}$

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\,dy\,dx$$

$$\int_0^1 x\int_0^{\sqrt{1-x^2}} y(1-y^2)^{-1/2}\,dy\,dx$$

$$-\frac{1}{2}\int_0^1 x\int_0^{\sqrt{1-x^2}} -2y(1-y^2)^{-1/2}\,dy\,dx$$

$$-\frac{1}{2}\int_0^1 x\left[\frac{(1-y^2)^{-1/2+1}}{-\frac{1}{2}+1}\right]_0^{\sqrt{1-x^2}} dx$$

$$-\frac{1}{2}\int_0^1 x\left[(1-1+x^2)^{1/2}-1\right] dx$$

$$= \cancel{f} - \int_0^1 x(x-1)\,dx$$

$$= -\int_0^1 x^2 - x\,dx = -\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1$$

$$= \frac{-1}{3} + \frac{1}{2} = -\frac{2+3}{6} = \frac{1}{6}$$

(13) Test the convergence of following series

$$\sum \frac{4.7.\,----\,(3n+1)}{n!} x^n$$

Sol^n^ $a_{n+1} = \frac{4.7.10\,---\,(3n+4)}{n!} x^n$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)}{(3n+4)} \cdot \frac{1}{x} = \frac{1+\frac{1}{n}}{3+\frac{4}{n}} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1+0}{3+0} \cdot \frac{1}{x} = \frac{1}{3x}$$

By Ratio test, If $\frac{1}{3x} > 1$ i.e $x < \frac{1}{3}$ converges

If $x > \frac{1}{3}$ diverges

$x = \frac{1}{3} \Rightarrow$ Ratio test fails

$$\frac{a_n}{a_{n+1}} = \frac{(1+\frac{1}{n})}{(3+\frac{4}{n})} \cdot 3 = \frac{1+\frac{1}{n}}{3(1+\frac{4}{3n})} \cdot 3 = \left(1+\frac{1}{n}\right)\left(1+\frac{4}{3n}\right)^{-1}$$

$$= \left(1+\frac{1}{n}\right)\left(\cancel{1}\ 1 - \frac{4}{3n} + O\left(\frac{1}{n^2}\right)\right) = 1 - \frac{1}{3n} + O\left(\frac{1}{n^2}\right)$$

Comparing it with $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$

By Gauss test, $\mu = -\frac{1}{3} < 1$

series is divergent.

series converges for $x < \frac{1}{3}$ and

diverges for $x \geq \frac{1}{3}$

verify if the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal and hence find its inverse.

Solⁿ $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$AA^{-1} = \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

AA^{-1} = Identity matrix

Yes it is orthogonal matrix

Inverse

$|AA^{-1}| = 1(1-0) - 0 - 0$

$|AA^{-1}| = 1$

$(AA^{-1})^{-1} = \frac{adj.(AA^{-1})}{|A|}$

$$(AA^{-1})^{-1} = \frac{1}{1}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}' = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Q) Find maximum and minimum value of

$x^3 - y^3 - 3axy.$

$$f = x^3 - y^3 - 3axy.$$

$$\frac{\partial f}{\partial x} = 3x^2 + 0 - 3ay = 3x^2 - 3ay.$$

$$\frac{\partial f}{\partial y} = 0 + 3y^2 - 3ax = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y}(3x^2 - 3ay)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 - 3a = -3a = s$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x \quad , t = \frac{\partial^2 f}{\partial y^2} = 6y.$$

$$r = 6x, \; s = -3a, \; t = 6y.$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay = 0$$

$$x^2 - ay = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0$$

$$y^2 - ax = 0 \quad \text{--- (2)}$$

from ① & ②

$x^2 - ay = 0 \Rightarrow \left(\frac{y^2}{a}\right)^2 - ay = 0 \quad \left[x = \frac{y^2}{a}\right]$

$y^4 - a^3 y = 0$

$y(y^3 - a^3) = 0$

$y = 0, a.$

$y = 0, x = 0$ or $y = a, x = \pm a$

$x = -a, y = a$ donot satisfy equation ②

At $x = 0, y = 0$

$r = 0, s = -3a, t = 0$

$rt - s^2 = 0 - (-3a)^2 = -ve$

neither max nor minimum at $x = 0, y = 0$

At $x = a, y = a$

$r = 6a, s = -3a, t = 6a$

$rt - s^2 = (6a)(6a) - (-3a)^2$

$= 36a^2 - 9a^2 > 0$

$r = 6a > 0$ if $a > 0$

$r = 6a < 0$ if $a < 0$

It is max. when $a > 0$ &

min when $a < 0$

(16) Solve the eq. $x+y+z=3$, $x+2y+3z=4$

$x+4y+9z=6$

Sol^n

$$x+y+z=3$$
$$x+2y+3z=4$$
$$x+4y+9z=6$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

Applying Row operation

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$5z=-3$, $\cancel{y+2}$ $y+2z=1$, $x+y+z=3$

$z=-3$, $y=\frac{11}{5}$, $z=-\frac{3}{5}$

(b) Find the inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

$$|A| = 2(12-2)-3(16-1)+4(8-3)$$
$$=2(10)-3(15)+4(5)$$
$$=20-45+20=40-45=-5$$

$$A^{-1}=\frac{Adj A}{|A|}$$

the Equations $x+y+z=3$, $x+2y+3z=4$, $x+4y+9z=6$

Augmented matrix $[A:B]$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

$R_2 - R_1$, $R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$\rho(A) = \rho(A:B) = 3 =$ no of unknows

$\Rightarrow$ Unique solution is there

$\Rightarrow$ $x+y+z=3 \Rightarrow x=2$

$y+2z=1 \Rightarrow y=1$

$2z=0 \Rightarrow z=0$

(b) Find Inverse of matrix $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Sol: $A = IA$ (We will use Gauss-Jordan method)

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_2 \to R_2/5$
$R_3 \to R_3/7$

$$\rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 4/5 \\ -1 & 0 & 2 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/5 & 4/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$R_2 \to R_2 - 3R_3$
$R_1 \to R_1 - 4R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -4/5 & -19/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$R_1 \to R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix} A$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -2 & 4/5 & 9/5 \\ 3 & -4/5 & -14/5 \\ -1 & 1/5 & 6/5 \end{bmatrix}$$

17 (a) Find area of the surface of revolution generated by revolving $x = y^3$ for $y = 0$ to $y = 2$

Sol: Surface area about (y-axis)

$$\int_{y_1}^{y_2} 2\pi x \frac{ds}{dy}\, dy \qquad \text{where } \frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \qquad \frac{dx}{dy} = 3y^2$$

$$\frac{ds}{dy} = \sqrt{1 + 3y^2}$$

Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$\int_{-1}^{1}\int_{0}^{z}\left(\cancel{\frac{x^2}{2}+xy+zx}\right)_{x-z}^{x+z}\cancel{dy}\,dz \Rightarrow \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$\int_{-1}^{1}\int_{0}^{z}\left(xy+\frac{y^2}{2}+zy\right)_{x-z}^{x+z}dx\;dz$

$\int_{-1}^{1}\int_{0}^{z}\left[x(x+z)+\frac{(x+z)^2}{2}+z(x+z)-x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right]dx\,dz$

$\int_{-1}^{1}\int_{0}^{z}\left(\cancel{x^2}+xz+\frac{x^2+z^2+2zx}{2}+\cancel{xz}+z^2-\cancel{x^2}+zx-\left(\frac{x^2+z^2+2zx}{2}\right)-\cancel{zx}+z^2\right)dx\,dz$

$\int_{-1}^{1}\int_{0}^{z}\left(2zx+\frac{4zx}{2}+2z^2\right)dx\,dz = \int_{-1}^{1}\int_{0}^{z}(4zx+2z^2)\,dx\,dz$

$\Rightarrow \int_{-1}^{1}\left(\frac{4zx^2}{2}+2z^2x\right)_{0}^{z}dz$

$\Rightarrow \int_{-1}^{1}\left(\frac{4z^3}{2}+2z^3\right)dz = \int_{-1}^{1}(2z^3+2z^3)\,dz$

$= \int_{-1}^{1}4z^3\,dz$

$= \left[\frac{4z^4}{4}\right]_{-1}^{1} \Rightarrow [z^4]_{-1}^{1}$

$\Rightarrow 1-1=0$

Surface area $S = 2\pi \int_0^2 y^3 \sqrt{1+3y^2}\, dy.$

Put $1+3y^2 = t$

$3(2y)dy = dt$

$y\,dy = \frac{dt}{6}$

$y^2 = \frac{t-1}{3}$

at $y=0 \Rightarrow t=1$

$y=2 \Rightarrow t=13$

$$S = 2\pi \int_0^2 y^2 \cdot y \sqrt{1+3y^2}\, dy$$

$$= 2\pi \int_1^{13} \frac{t-1}{3} \sqrt{t} \frac{dt}{6}$$

$$= \frac{2\pi}{18} \int_1^{13} \sqrt{t}\,(t-1)\, dt$$

$$= \frac{\pi}{9} \int_1^{13} (t^{3/2} - t^{1/2}) dt$$

$$\frac{\pi}{9} \left[\frac{t^{5/2}}{5/2} - \frac{t^{3/2}}{3/2} \right]_1^{13}$$

$$\frac{2\pi}{9} \left[\frac{t^{5/2}}{5} - \frac{t^{3/2}}{3} \right]_1^{13}$$

$$\frac{2\pi}{9} \left[\frac{(13)^{5/2}}{5} - \frac{(13)^{3/2}}{3} - \frac{1}{5} + \frac{1}{3} \right]$$

$$\frac{2\pi}{9} \left[(13)^{3/2} \left[\frac{13}{5} - \frac{1}{3} \right] - 2/15 \right]$$

$$\frac{2\pi}{9} \left[(13)^{3/2} \left(\frac{34}{15}\right) - \left(\frac{2}{15}\right) \right]$$

$$\frac{2\pi}{9 \times 15} \left[(34)\,(13)^{3/2} - 2 \right]$$

$$\frac{2\pi}{135} \left(34 \times 13 \times \sqrt{13} - 2 \right)$$

$$\frac{2\pi}{135} \left(442\sqrt{13} - 2 \right)$$ Ans:-

Test the convergence of series

$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}}$

Let $\sum a_n = \frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$

$$= \sum_{n=1}^{\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

Here $a_n = \frac{x^{2n-2}}{\sqrt{n}(n+1)}$

$a_{n+1} = \frac{x^{2n}}{\sqrt{n+1}\,(n+2)}$

$$\frac{a_n}{a_{n+1}} = \frac{x^{2n-2}}{x^{2n}} \cdot \frac{\sqrt{n+1}\,(n+2)}{\sqrt{n}\,(n+1)}$$

$$= \frac{1}{x^2}\sqrt{\frac{n+1}{n}}\left(\frac{n+2}{n+1}\right)$$

$$= \frac{1}{x^2}\sqrt{1+1/n}\left(\frac{1+2/n}{1+1/n}\right)$$

By Ratio test $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1}{x^2}\sqrt{1+1/n}\left(\frac{1+2/n}{1+1/n}\right)$

$$= \frac{1}{x^2}$$

$\sum a_n$ will converge if $\frac{1}{x^2} > 1 \Rightarrow x^2 < 1$

$\sum a_n$ will diverge if $\frac{1}{x^2} < 1 \Rightarrow x^2 > 1$

for $x^2 = 1$; Ratio test fails

Now $a_n = \frac{b}{x^2}$

$$a_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

for $x^2 = 1$ $\Rightarrow$ $a_n = \frac{(x^2)^{n-1}}{(n+1)\sqrt{n}} = \frac{1}{\sqrt{n}(n+1)}$

By Comparison test $\Rightarrow$ $a_n = \frac{1}{\sqrt{n}.\,n(1+1/n)}$

Select $b_n = \frac{1}{n^{3/2}}$

Now $\frac{a_n}{b_n} = \frac{1}{1+1/n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (finite $\neq 0$)

$\Rightarrow$ Σa_n and Σb_n will behave alike

as $\Sigma b_n = \Sigma \frac{1}{n^{3/2}}$ Here $p = 3/2 > 1 \Rightarrow$ cgt

$\Rightarrow$ Σa_n will be cgt

thus Σa_n is Convergent for $x^2 \leq 1$
divergent for $x^2 > 1$

(b) Find Maclaurin series for $f(x) = \cos x$

Sol: Here $f(x) = \cos x$ and $x \in [0, x]$

By Tayl Maclaurin series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = \cos 0 = 1$

$\vdots$

Put in (1)

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$$= 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$